

# Complemented subspaces of homogeneous polynomials

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## Abstract

Let  $\mathcal{P}_K(^nE; F)$  (resp.  $\mathcal{P}_w(^nE; F)$ ) denote the subspace of all  $P \in \mathcal{P}(^nE; F)$  which are compact (resp. weakly continuous on bounded sets). We show that if  $\mathcal{P}_K(^nE; F)$  contains an isomorphic copy of  $c_0$ , then  $\mathcal{P}_K(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$ . Likewise we show that if  $\mathcal{P}_w(^nE; F)$  contains an isomorphic copy of  $c_0$ , then  $\mathcal{P}_w(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$ .

**Keywords:** Banach space, linear operator, compact operator, homogeneous polynomial, complemented subspace, unconditional basis.

## 1 Introduction

The problem of establishing sufficient conditions for the complementation of the subspace of compact linear operators  $\mathcal{L}_K(E; F)$  in the space  $\mathcal{L}(E; F)$  of all continuous linear operators, has been widely studied by many authors. For example, see Kalton [18], Emmanuelle [12], John [17], Bator and Lewis [5] and Ghenciu [14], among others.

Emmanuelle [12] and John [17] showed that if  $c_0$  embeds in  $\mathcal{L}_K(E; F)$  then  $\mathcal{L}_K(E; F)$  is not complemented in  $\mathcal{L}(E; F)$  for every  $E$  and  $F$  infinite dimensional Banach spaces.

John [17] proved that if  $E$  and  $F$  are arbitrary Banach spaces and  $T : E \rightarrow F$  is a non compact operator which admits a factorization  $T = A \circ B$  through a Banach space  $G$  with an unconditional basis, then the subspace  $\mathcal{L}_K(E; F)$  of compact operators contains an isomorphic copy of  $c_0$  and thus  $\mathcal{L}_K(E; F)$  is not complemented in  $\mathcal{L}(E; F)$ . John [17] also proved that if  $E$  and  $F$  are infinite dimensional Banach spaces, such that each non compact operator  $T \in \mathcal{L}(E; F)$  factors through a Banach space  $G$  with an unconditional basis, then the following conditions are equivalent:

1.  $\mathcal{L}_K(E; F) = \mathcal{L}(E; F)$ .
2.  $\mathcal{L}(E; F)$  contains no copy of  $\ell_\infty$ .
3.  $\mathcal{L}_K(E; F)$  contains no copy of  $c_0$ .
4.  $\mathcal{L}_K(E; F)$  is complemented in  $\mathcal{L}(E; F)$ .

Ghenciu [14] obtained the following result: Let  $E$  and  $F$  be Banach spaces, and let  $G$  be a Banach space with an unconditional basis  $(g_n)$  and coordinate functionals  $(g'_n)$ .

- (a) If there exist operators  $R \in \mathcal{L}(G; F)$  and  $S \in \mathcal{L}(E; G)$  such that  $(R(g_n))$  is a seminormalized basic sequence in  $F$  and  $(S'(g'_n))$  is not relatively compact in  $E'$ , then  $\mathcal{L}_K(E; F)$  is not complemented in  $\mathcal{L}(E; F)$ .

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- (b) If there exist operators  $R \in \mathcal{L}(G; F)$  and  $S \in \mathcal{L}(E; G)$  such that  $(R(g_n))$  is a seminormalized basic sequence in  $F$  and  $(S'(g'_n))$  is not relatively weakly compact in  $E'$ , then  $\mathcal{L}_{wK}(E; F)$  is not complemented in  $\mathcal{L}(E; F)$ .

This result generalizes results of several authors [11],[5], [13]. In this paper, we obtain polynomial versions of the preceding results.

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## 2 Preliminaries

Let  $E$  and  $F$  denote Banach spaces over  $\mathbb{K}$ , where  $\mathbb{K}$  is  $\mathbb{R}$  or  $\mathbb{C}$ . Let  $E'$  denote the dual of  $E$ . Denote by  $\mathcal{L}(E; F)$ ,  $\mathcal{L}_K(E; F)$  and  $\mathcal{L}_{wK}(E; F)$ , respectively, the spaces of all bounded, all compact and all weakly compact linear operators of  $E$  into  $F$ . Let  $\mathcal{P}({}^n E; F)$  denote the Banach space of all continuous  $n$ -homogeneous polynomials from  $E$  into  $F$ . We omit  $F$  when  $F = \mathbb{K}$ . Let  $\mathcal{P}_w({}^n E; F)$  denote the subspace of all  $P \in \mathcal{P}({}^n E; F)$  which are weakly continuous on bounded sets, that is the restriction  $P|_B : B \rightarrow F$  is continuous for each bounded set  $B \subset E$ , when  $B$  and  $F$  are endowed with the weak topology and the norm topology, respectively. Let  $\mathcal{P}_K({}^n E; F)$  denote the subspace of all  $P \in \mathcal{P}({}^n E; F)$  which map bounded sets onto relatively compact sets. Let  $\mathcal{P}_{wK}({}^n E; F)$  denote the subspace of all  $P \in \mathcal{P}({}^n E; F)$  which map bounded sets onto relatively weakly compact sets. We always have the inclusions

$$\mathcal{P}_w({}^n E; F) \subset \mathcal{P}_K({}^n E; F) \subset \mathcal{P}_{wK}({}^n E; F) \subset \mathcal{P}({}^n E; F).$$

We refer to [10] or [19] for background information on the theory of polynomials on Banach spaces.

$E$  is isomorphic to a complemented subspace of  $F$  if and only if there are  $A \in \mathcal{L}(E; F)$  and  $B \in \mathcal{L}(F; E)$  such that  $B \circ A = I$ .  $E$  is said to have an unconditional finite dimensional expansion of the identity if there is a sequence of bounded linear operators  $A_n : E \rightarrow E$  of finite rank, such that for  $x \in E$

$$\sum_{n=1}^{\infty} A_n(x) = x$$

unconditionally.

We will say that the series  $\sum_{n=1}^{\infty} x_n$  of elements of  $X$  is weakly unconditionally Cauchy if  $\sum_{n=1}^{\infty} |x'(x_n)| < \infty$  for all  $x' \in X'$  or, equivalently if

$$\sup \left\{ \left\| \sum_{n \in F} x_n \right\|; F \subset \mathbb{N}, F \text{ finite} \right\} < \infty.$$

A sequence  $(x_n) \subset E$  is a semi-normalized basic sequence if  $(x_n)$  is a Schauder basis for the closed subspace  $M = \overline{[x_n : n \in \mathbb{N}]}$ , and moreover there are constant  $a$  and  $b$  such that  $0 < a < \|x_n\| < b$  for all  $n \in \mathbb{N}$ . We denote by  $(e_n)$  the canonical basis of  $c_0$ . If  $\Sigma$  is an algebra of subsets of a set  $\Omega$ , then a finitely additive vector measure

$\mu : \Sigma \rightarrow E$  is said to be strongly additive if the series  $\sum_{n=1}^{\infty} \mu(A_n)$  converges in norm for each sequence  $(A_n)$  of pairwise disjoint members of  $\Sigma$ . The Diestel-Faires theorem (see [9, p.20, Theorem 2]) asserts that if  $\Sigma$  is a  $\sigma$ -algebra and  $\mu : \Sigma \rightarrow E$  is not strongly additive, then  $E$  contains an isomorphic copy of  $\ell_{\infty}$ .

## 3 The main results

The proof of our main results rests mainly on the following theorem of Ghenciu [14], which generalizes results of several authors [11],[5], [13].

**Theorem 3.1.** ([14, Theorem 1]) Let  $E$  and  $F$  be Banach spaces, and let  $G$  be a Banach space with an unconditional basis  $(g_n)$  and coordinate functionals  $(g'_n)$ .

- (a) If there exist operators  $R \in \mathcal{L}(G; F)$  and  $S \in \mathcal{L}(E; G)$  such that  $(R(g_n))$  is a seminormalized basic sequence in  $F$  and  $(S'(g'_n))$  is not relatively compact in  $E'$ , then  $\mathcal{L}_K(E; F)$  is not complemented in  $\mathcal{L}(E; F)$ .
- (b) If there exist operators  $R \in \mathcal{L}(G; F)$  and  $S \in \mathcal{L}(E; G)$  such that  $(R(g_n))$  is a seminormalized basic sequence in  $F$  and  $(S'(g'_n))$  is not relatively weakly compact in  $E'$ , then  $\mathcal{L}_{wK}(E; F)$  is not complemented in  $\mathcal{L}(E; F)$ .

Emmanuele [12] and John [17] independently proved that if  $\mathcal{L}_K(E; F)$  contains a copy of  $c_0$ , then  $\mathcal{L}_K(E; F)$  is not complemented in  $\mathcal{L}(E; F)$  (see [12, Theorem 2] and [17, Theorem 1]). They also proved that if there exists a noncompact operator  $T \in \mathcal{L}(E; F)$  which factors through a Banach space with an unconditional basis, then  $\mathcal{L}_K(E; F)$  contains a copy of  $c_0$ . Clearly Theorem 3.1 (a) follows from these results.

**Theorem 3.2.** Let  $E$  and  $F$  be Banach spaces, and let  $G$  be a Banach space with an unconditional basis  $(g_n)$  and coordinate functionals  $(g'_n)$ .

- (a) If there exist operators  $R \in \mathcal{L}(G; F)$  and  $S \in \mathcal{L}(E; G)$  such that  $(R(g_n))$  is a seminormalized basic sequence in  $F$  and  $(S'(g'_n))$  is not relatively compact in  $E'$ , then  $\mathcal{P}_K(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .
- (b) If there exist operators  $R \in \mathcal{L}(G; F)$  and  $S \in \mathcal{L}(E; G)$  such that  $(R(g_n))$  is a seminormalized basic sequence in  $F$  and  $(S'(g'_n))$  is not relatively weakly compact in  $E'$ , then  $\mathcal{P}_{wK}(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .

*Proof.* (a) The case  $n = 1$  follows from Theorem 3.1 (a). If  $n \in \mathbb{N}$ , then by a result of Ryan [21] there exists an isomorphism

$$P \in \mathcal{P}(^nE; F) \rightarrow T_P \in \mathcal{L}(\hat{\otimes}_{n,s,\pi} E; F).$$

Furthermore  $P \in \mathcal{P}_K(^nE; F)$  if and only if  $T_P \in \mathcal{L}_K(\hat{\otimes}_{n,s,\pi} E; F)$ . Suppose that  $\mathcal{P}_K(^nE; F)$  is complemented in  $\mathcal{P}(^nE; F)$ . Then  $\mathcal{L}_K(\hat{\otimes}_{n,s,\pi} E; F)$  is complemented in  $\mathcal{L}(\hat{\otimes}_{n,s,\pi} E; F)$ . Let  $\pi : \mathcal{L}(\hat{\otimes}_{n,s,\pi} E; F) \rightarrow \mathcal{L}_K(\hat{\otimes}_{n,s,\pi} E; F)$  be a projection. By a result of Blasco [7, Theorem 3]  $E$  is isomorphic to a complemented subspace of  $\hat{\otimes}_{n,s,\pi} E$ . Hence there exist operators  $A \in \mathcal{L}(E; \hat{\otimes}_{n,s,\pi} E)$  and  $B \in \mathcal{L}(\hat{\otimes}_{n,s,\pi} E; E)$  such that  $B \circ A = I$ . Consider the operator

$$\rho : T \in \mathcal{L}(E; F) \rightarrow \pi(T \circ B) \circ A \in \mathcal{L}_K(E; F).$$

If  $T \in \mathcal{L}_K(E; F)$ , then  $T \circ B \in \mathcal{L}_K(\hat{\otimes}_{n,s,\pi} E; F)$  and therefore  $\pi(T \circ B) \circ A = T \circ B \circ A = T$ . Thus  $\rho : \mathcal{L}(E; F) \rightarrow \mathcal{L}_K(E; F)$  is a projection, contradicting the case  $n = 1$ .

(b) The proof of (b) is almost identical to the proof of (a), but using that  $P \in \mathcal{P}_{wK}(^nE; F)$  if and only if  $T_P \in \mathcal{L}_{wK}(\hat{\otimes}_{n,s,\pi} E; F)$ , a result which is also due to Ryan [21].  $\square$

**Theorem 3.3.** Let  $E$  and  $F$  be Banach spaces, and let  $G$  be a Banach space with an unconditional basis  $(g_n)$  and coordinate functionals  $(g'_n)$ . If there exist operators  $R \in \mathcal{L}(G; F)$  and  $S \in \mathcal{L}(E; G)$  such that  $(R(g_n))$  is a seminormalized basic sequence in  $F$  and  $(S'(g'_n))$  is not relatively compact in  $E'$ , then  $\mathcal{P}_w(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .

*Proof.* The method of proof of Theorem 3.2 does not work here, since it is not true in general that  $P \in \mathcal{P}_w(^nE; F)$  if and only if  $T_P \in \mathcal{L}_w(\hat{\otimes}_{n,s,\pi} E; F)$ . Thus we have to proceed differently. It follows from results of Aron and Prolla [3] and Aron, Hervés and Valdivia [2] that  $\mathcal{P}_w(^nE; F) \subset \mathcal{P}_K(^nE; F)$  for every  $n \in \mathbb{N}$ , and it is easy to

see that  $\mathcal{P}_w(^nE; F) = \mathcal{P}_K(^nE; F)$  when  $n = 1$ . Thus the case  $n = 1$  follows from Theorem 3.1 (a). To prove the theorem by induction on  $n$  it suffices to prove that if  $\mathcal{P}_w(^{n+1}E; F)$  is complemented in  $\mathcal{P}(^{n+1}E; F)$ , then  $\mathcal{P}_w(^nE; F)$  is complemented in  $\mathcal{P}(^nE; F)$ . Aron and Schottenloher [4, Proposition 5.3] proved that  $\mathcal{P}(^nE; F)$  is isomorphic to a complemented subspace of  $\mathcal{P}(^{n+1}E; F)$  when  $F$  is the scalar field, but their proof works equally well when  $F$  is an arbitrary Banach space. Thus there exist operators  $A \in \mathcal{L}(\mathcal{P}(^nE; F); \mathcal{P}(^{n+1}E; F))$  and  $B \in \mathcal{L}(\mathcal{P}(^{n+1}E; F); \mathcal{P}(^nE; F))$  such that  $B \circ A = I$ . The operator  $A$  is of the form

$$A(P)(x) = \varphi_0(x)P(x)$$

for every  $P \in \mathcal{P}(^nE; F)$  and  $x \in E$ , where  $\varphi_0 \in E'$  verifies that  $\|\varphi_0\| = 1 = \varphi_0(x_0)$ , where  $x_0 \in E$  and  $\|x_0\| = 1$ . It is clear that if  $P \in \mathcal{P}_w(^nE; F)$ , then  $A(P) \in \mathcal{P}_w(^{n+1}E; F)$ . Let us assume that  $\mathcal{P}_w(^{n+1}E; F)$  is complemented in  $\mathcal{P}(^{n+1}E; F)$ , and let  $\pi : \mathcal{P}(^{n+1}E; F) \rightarrow \mathcal{P}_w(^{n+1}E; F)$  be a projection. Consider the operator

$$\rho = B \circ \pi \circ A : \mathcal{P}(^nE; F) \rightarrow \mathcal{P}_w(^nE; F).$$

If  $P \in \mathcal{P}_w(^nE; F)$ , then  $A(P) \in \mathcal{P}_w(^{n+1}E; F)$ , and therefore

$$\rho(P) = B \circ \pi \circ A(P) = B \circ A(P) = P.$$

Thus  $\rho : \mathcal{P}(^nE; F) \rightarrow \mathcal{P}_w(^nE; F)$  is a projection, and therefore  $\mathcal{P}_w(^nE; F)$  is complemented in  $\mathcal{P}(^nE; F)$ . This completes the proof.  $\square$

Ghenciu [14] derived as corollaries of Theorem 3.1 results of several authors [11], [5], [13], [18] and [17]. We now apply Theorems 3.2 and 3.3 to obtain polynomials versions of those corollaries.

**Corollary 3.4.** *If  $F$  contains a copy of  $c_0$  and  $E'$  contains a weak-star null sequence which is not weakly null, then  $\mathcal{P}_{wK}(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .*

**Corollary 3.5.** *If  $F$  contains a copy of  $c_0$  and  $E$  contains a complemented copy of  $c_0$ , then  $\mathcal{P}_{wK}(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .*

**Corollary 3.6.** *If  $F$  contains a copy of  $\ell_1$  and  $\mathcal{L}(E; \ell_1) \neq \mathcal{L}_K(E; \ell_1)$ , then  $\mathcal{P}_{wK}(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .*

When  $n = 1$  Corollaries 3.4, 3.5 and 3.6 correspond to [14, Corollaries 2,3 and 5]. Ghenciu derived those corollaries by observing that  $E$  and  $F$  satisfy the hypothesis of Theorem 3.1 (b). Since the hypothesis of Theorem 3.1 (b) coincide with the hypothesis of Theorem 3.2 (b), we see that Corollaries 3.4, 3.5 and 3.6 follow from Theorem 3.2 (b).

**Corollary 3.7.** *If  $F$  contains a copy of  $c_0$  and  $E$  is infinite dimensional, then:*

- (a)  $\mathcal{P}_K(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .
- (b)  $\mathcal{P}_w(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .

**Corollary 3.8.** *If  $E$  contains a complemented copy of  $\ell_1$  and  $F$  is infinite dimensional, then:*

- (a)  $\mathcal{P}_K(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .
- (b)  $\mathcal{P}_w(^nE; F)$  is not complemented in  $\mathcal{P}(^nE; F)$  for every  $n \in \mathbb{N}$ .

When  $n = 1$  Corollaries 3.7 and 3.8 correspond to [14, Corollaries 4 and 6]. Ghenciu derived those corollaries by observing that  $E$  and  $F$  satisfy the hypothesis of Theorem 3.1 (a). Since the hypothesis of Theorem 3.1 (a) coincide with the hypothesis of Theorems 3.2 (a) and 3.3, we see that Corollaries 3.7 and 3.8 follow from Theorems 3.2 (a) and 3.3.

**Corollary 3.9.** *If  $E$  contains a copy of  $\ell_1$  and  $F$  contains a copy of  $\ell_p$ , with  $2 \leq p < \infty$ , then:*

- (a)  $\mathcal{P}_K(^n E; F)$  is not complemented in  $\mathcal{P}(^n E; F)$  for every  $n \in \mathbb{N}$ .
- (b)  $\mathcal{P}_w(^n E; F)$  is not complemented in  $\mathcal{P}(^n E; F)$  for every  $n \in \mathbb{N}$ .

*Proof.* We follow an argument of Emmanuele [12, p. 334]. By a result of Pelczynski [20], if  $E$  contains a copy of  $\ell_1$ , then  $E$  has a quotient isomorphic to  $\ell_2$  (see also the proof of [1]). Let  $S : E \rightarrow \ell_2$  be the quotient mapping, and let  $R : \ell_2 \hookrightarrow \ell_p \subset F$  be the natural inclusion. Since  $S' : \ell_2 \rightarrow E'$  is an embedding, the hypothesis of Theorems 3.2 (a) and 3.3 are clearly satisfied.  $\square$

**Proposition 3.10.** *Let  $E$  and  $F$  be infinite dimensional Banach spaces. If  $\mathcal{P}_K(^n E; F)$  contains a copy of  $c_0$ , then  $\mathcal{P}_K(^n E; F)$  is not complemented in  $\mathcal{P}(^n E; F)$ .*

*Proof.* By an aforementioned result of Ryan [21] we have that  $P \in \mathcal{P}_K(^n E; F)$  if and only if  $T_P \in \mathcal{L}_K(\hat{\otimes}_{n,s,\pi} E; F)$ . Thus the result follows from [12, Theorem 2] or [17, Theorem 1].  $\square$

The next proposition is a polynomial version of [12, Theorem 2] and [17, Theorem 1]. The proof is based in ideas of [15, Corollary 11].

**Proposition 3.11.** *Let  $E$  be an infinite dimensional Banach space and  $n > 1$ . If  $\mathcal{P}_w(^n E; F)$  contains a copy of  $c_0$ , then  $\mathcal{P}_w(^n E; F)$  is not complemented in  $\mathcal{P}(^n E; F)$ .*

*Proof.* By Corollary 3.7 and [16, Lemma 5] we may suppose without loss of generality that  $F$  contains no copy of  $c_0$  and  $E$  contains no complemented copy of  $\ell_1$ . By [16, Theorem 3]  $\mathcal{P}_w(^n E; F)$  contains no copy of  $\ell_\infty$ . Let  $(P_i)$  be a copy of the unit vector basis  $(e_i)$  of  $c_0$  in  $\mathcal{P}_w(^n E; F)$ . Then

$$\sup \left\{ \left\| \sum_{i \in F} e_i \right\| ; F \subset \mathbb{N}, F \text{ finite} \right\} = 1.$$

By a result of Bessaga and Pelczynski [6] (see also [8, p.44, Theorem 6]) the series  $\sum_{i=1}^{\infty} e_i$  is weakly unconditionally

Cauchy in  $c_0$ . This implies that the series  $\sum_{i=1}^{\infty} P_i$  is weakly unconditionally Cauchy in  $\mathcal{P}_w(^n E; F)$ . For every  $\varphi \in F'$  and  $x \in E$  we consider the continuous linear functional

$$\psi : P \in \mathcal{P}_w(^n E; F) \rightarrow \varphi(P(x)) \in \mathbb{C}.$$

Since the series  $\sum_{i=1}^{\infty} P_i$  is weakly unconditionally Cauchy in  $\mathcal{P}_w(^n E; F)$ ,  $\sum_{i=1}^{\infty} |\psi(P_i)| = \sum_{i=1}^{\infty} |\varphi(P_i(x))| < \infty$  for

every  $\varphi \in F'$  and  $x \in E$ . This shows that  $\sum_{i=1}^{\infty} P_i(x)$  is weakly unconditionally Cauchy in  $F$  for each  $x \in E$ .

Finally since  $F$  contains no copy of  $c_0$ , an application of [8, p.45, Theorem 8] shows that  $\sum_{i=1}^{\infty} P_i(x)$  converges

unconditionally in  $F$  for each  $x \in E$ . Let  $\mu : \wp(\mathbb{N}) \rightarrow \mathcal{P}(^n E; F)$  be the finitely additive vector measure defined by  $\mu(A)(x) = \sum_{i \in A} P_i(x)$  for each  $x \in E$  and  $A \subset \mathbb{N}$ . Suppose there is a projection  $\pi : \mathcal{P}(^n E; F) \rightarrow \mathcal{P}_w(^n E; F)$ .

Then  $\pi(P_i) = P_i$  for each  $i \in \mathbb{N}$ . If the sequence  $(\|P_i\|)$  does not converge to zero, then there is  $\epsilon > 0$  and a subsequence  $(i_k)$  of  $\mathbb{N}$ , such that  $\|P_{i_k}\| > \epsilon$  for each  $k \in \mathbb{N}$ . But this implies that the measure  $\pi \circ \mu : \wp(\mathbb{N}) \rightarrow \mathcal{P}_w(^n E; F)$  is not strongly additive. Then the Diestel-Faires Theorem would imply that  $\mathcal{P}_w(^n E; F)$  contains a copy of  $\ell_\infty$ . Therefore  $\|P_i\| \rightarrow 0$ , but this is absurd too, because  $(P_i)$  is a copy of  $(e_i)$ . This complete the proof.  $\square$

The following theorem is a polynomial version of [17, Theorem 2 ].

**Theorem 3.12.** *Let  $E$  and  $F$  be Banach spaces and  $P \in \mathcal{P}(^n E; F)$  such that  $P \notin \mathcal{P}_w(^n E; F)$ . Suppose that  $P$  admits a factorization  $P = Q \circ T$  through a Banach space  $G$  with an unconditional finite dimensional expansion of the identity, where  $T \in \mathcal{L}(E; G)$  and  $Q \in \mathcal{P}(^n G; F)$ . Then  $\mathcal{P}_w(^n E; F)$  contains a copy of  $c_0$  and thus  $\mathcal{P}_w(^n E; F)$  is not complemented in  $\mathcal{P}(^n E; F)$ .*

*Proof.* The case  $n = 1$  follows from [17, Theorem 2 ].

Case  $n > 1$ : Since  $G$  has an unconditional finite dimensional expansion of the identity, by [16, Lemma 6 ] there is a sequence  $(Q_i) \subset \mathcal{P}_w(^n G; F)$  so that  $Q(z) = \sum_{i=1}^{\infty} Q_i(z)$  unconditionally for each  $z \in G$ , hence  $P(x) = \sum_{i=1}^{\infty} Q_i(T(x))$  unconditionally for each  $x \in E$ . Since  $Q_i \in \mathcal{P}_w(^n G; F)$  for every  $i \in \mathbb{N}$ , it follows that  $Q_i \circ T \in \mathcal{P}_w(^n E; F)$  for every  $i \in \mathbb{N}$ . By the uniform boundedness principle, we have

$$\sup \left\{ \left\| \sum_{i \in F} Q_i \circ T \right\|; F \subset \mathbb{N}, F \text{ finite} \right\} < \infty.$$

Again by [8, p.44, Theorem 6] the series  $\sum_{i=1}^{\infty} Q_i \circ T$  is weakly unconditionally Cauchy in  $\mathcal{P}_w(^n E; F)$ . Since  $P \notin \mathcal{P}_w(^n E; F)$ , an application of [8, p.45, Theorem 8] shows that  $\mathcal{P}_w(^n E; F)$  contains a copy of  $c_0$ , and therefore by Proposition 3.11  $\mathcal{P}_w(^n E; F)$  is not complemented in  $\mathcal{P}(^n E; F)$ .  $\square$

**Corollary 3.13.** *Let  $E$  and  $F$  be Banach spaces, with  $E$  infinite dimensional, and let  $n > 1$ . If each  $P \in \mathcal{P}(^n E; F)$  such that  $P \notin \mathcal{P}_w(^n E; F)$  admits a factorization  $P = Q \circ T$ , where  $T \in \mathcal{L}(E; G)$ ,  $Q \in \mathcal{P}(^n G; F)$  and  $G$  is a Banach space with an unconditional finite dimensional expansion of the identity, then the following conditions are equivalent:*

- (1)  $\mathcal{P}_w(^n E; F)$  contains a copy of  $c_0$ ,
- (1')  $\mathcal{P}_K(^n E; F)$  contains a copy of  $c_0$ ,
- (2)  $\mathcal{P}_w(^n E; F)$  is not complemented in  $\mathcal{P}(^n E; F)$ ,
- (2')  $\mathcal{P}_K(^n E; F)$  is not complemented in  $\mathcal{P}(^n E; F)$ ,
- (3)  $\mathcal{P}_w(^n E; F) \neq \mathcal{P}(^n E; F)$ ,
- (3')  $\mathcal{P}_K(^n E; F) \neq \mathcal{P}(^n E; F)$ ,
- (4)  $\mathcal{P}(^n E; F)$  contains a copy of  $c_0$ ,
- (5)  $\mathcal{P}(^n E; F)$  contains a copy of  $\ell_{\infty}$ .

*Proof.* (1)  $\Rightarrow$  (2) by Proposition 3.11.

(2)  $\Rightarrow$  (3) is obvious.

(3)  $\Rightarrow$  (1) by Theorem 3.12.

(1)  $\Rightarrow$  (4) is obvious.

(4)  $\Rightarrow$  (3) suppose (4) holds and (3) does not hold. Then  $\mathcal{P}_w(^n E; F) = \mathcal{P}(^n E; F) \supset c_0$ . Thus (1) holds, and therefore (3) holds, a contradiction.

(5)  $\Rightarrow$  (4) is obvious.



(4)  $\Rightarrow$  (5) by a result of Ryan [21]  $\mathcal{P}(^n E; F)$  is isometrically isomorphic to  $\mathcal{L}(\widehat{\otimes}_{n,s,\pi} E; F)$ . Thus the result follows from ([17, Remark 3 e] part 2  $\Rightarrow$  3).

Thus (1), (2), (3), (4) and (5) are equivalent.

(1)  $\Rightarrow$  (1') is obvious.

(1')  $\Rightarrow$  (2') by Proposition 3.10.

(2')  $\Rightarrow$  (3') is obvious.

(3')  $\Rightarrow$  (3) is obvious.

Since (3)  $\Rightarrow$  (1) and (1)  $\Rightarrow$  (1'), the proof of the corollary is complete.  $\square$

In particular if  $E$  has an unconditional finite dimensional expansion of the identity we obtain [16, Theorem 7]. The assumptions of this corollary apply also if  $F$  is a complemented subspace of a space with an unconditional basis.

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